

# Weak invariance principle for the local times of partial sums of Markov Chains

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**Abstract** Let  $\{X_n\}$  be an integer valued Markov Chain with finite state space. Let  $S_n = \sum_{k=0}^n X_k$  and let  $L_n(x)$  be the number of times  $S_k$  hits  $x \in \mathbb{Z}$  up to step  $n$ . Define the normalized local time process  $l_n(t, x)$  by

$$l_n(t, x) = \frac{L_{[nt]}(\lfloor \sqrt{n}x \rfloor)}{\sqrt{n}}, \quad x \in \mathbb{R}.$$

The subject of this paper is to prove a functional, weak invariance principle for the normalized sequence  $l_n(t, x)$ , i.e. we prove under the assumption of strong aperiodicity of the Markov Chain that the normalized local times converge in distribution to the local time of the Brownian Motion.

## 1 Introduction.

Let  $S \subseteq \mathbb{Z}$  be a finite set,  $P : S \times S \rightarrow [0, 1]$  an irreducible, aperiodic stochastic matrix. Let  $\mu$  be some distribution on  $S$  and let  $\{X_i\}_{i=0}^\infty$  be a Markov Chain generated by  $P$  and the initial distribution  $\mu$ , i.e.  $X_i, i = 0, 1, 2, \dots$  are random variables defined on a probability space  $(X, \mathcal{B}, P^\mu)$ , taking values in  $S$ , such that the equality

$$\mathbb{P}^\mu(X_{i_1} = s_1, X_{i_2} = s_2, \dots, X_{i_n} = s_n) = \sum_{s_0 \in S} \mu_{s_0} \cdot P^{i_1}(s_0, s_1) \cdot P^{i_2 - i_1}(s_1, s_2) \dots \cdot P^{i_n - i_{n-1}}(s_{n-1}, s_n),$$

holds with  $\mu_s = \mu\{s\}$ . Under these assumptions,  $P$  has a unique  $P$ -invariant probability distribution vector  $\nu$ , in the sense that the equality  $\nu P = \nu$  holds.

We may assume that for every initial distribution  $\mu$ ,  $X = S^\mathbb{N}$ ,  $\mathcal{B}$  is the Borel  $\sigma$ -field generated by cylinders of the type  $[s_0 s_1 \dots s_n] = \{\omega \in S^\mathbb{N} : \omega_i = s_i, i = 0, 1, \dots, n\}$  and  $X_i(\omega) = \omega_i$ .

Set

$$S_n = \sum_{k=0}^n X_k, \quad W_n(t) = \frac{1}{\sqrt{n}} S_{[nt]}, \quad t \in [0, 1].$$

Let  $D_{[s,t]}$ ,  $D_+$  and  $D$  denote the spaces of functions defined on  $[s, t]$ ,  $[0, \infty)$  and  $\mathbb{R}$  respectively, that are continuous on the right with left limits at each point (Cadlag functions).  $D_{[s,t]}$ ,  $D_+$  and  $D$  endowed with the Skorokhod  $J_1$  topology and its suitable generalization to  $[0, \infty)$  and  $\mathbb{R}$ , constitute Polish (complete and separable metrizable) spaces (see [3]).

Under the assumption that  $E^\nu(X_0) = 0$ , the weak invariance principle for the sequence  $W_n$  holds for any initial distribution  $\mu$ , i.e. the process  $W_n(t)$  regarded as a sequence of random variables defined on  $(X, \mathcal{B}, P^\mu)$  and taking values in  $D_+$  converges in distribution to the Brownian motion, henceforth denoted by  $W_\sigma(t)$ , where

$$\sigma^2 = \text{Var}(W_\sigma(1)) = \lim_{n \rightarrow \infty} \frac{E^\nu(S_n^2)}{n}$$

is the asymptotic variance of the process  $S_n$  with respect to the initial distribution  $\nu$ . Now,  $\sigma^2 = 0$  if and only if  $X$  is a coboundary, meaning that there exists a square integrable process  $Y = \{Y_n\}$  such that (see [12, Lemma 6] or [7, II.3 Thm.A])

$$X_n = Y_n - Y_{n+1}.$$

In this case the functional limit of  $W_n$  is degenerate, so we restrict our attention to the case when  $\sigma^2 > 0$ . The invariance principle may be proved by techniques used in this article, or otherwise, see [3].

Let  $L_n(x) = \#\{k \leq n : S_k = x\} = \sum_{k=0}^n \mathbf{1}_{\{x\}}(S_k)$  denote the number of arrivals of the process  $\{S_k\}_{k \in \mathbb{N}}$  at the point  $x$  after  $n$  steps, and let  $l_n(t, x) = \frac{1}{\sqrt{n}} L_{[nt]}([\sqrt{n}x])$ ,  $(t, x) \in [0, \infty) \times \mathbb{R}$ .  $l_n(t, x)$  is the amount of time multiplied by  $\sqrt{n}$  that the process  $W_n(\cdot)$  spends up to the time instant  $t$  at the point  $\frac{[\sqrt{n}x]}{\sqrt{n}}$ .

Let  $l(t, x)$  be the local time of the Brownian motion  $W_\sigma(\cdot)$  at time  $t$ , i.e.  $l(\cdot, \cdot)$  is a random function satisfying the equality

$$\int_A l(t, x) dx = \int_0^t \mathbf{1}_A(W_\sigma(t)) dt$$

for every Borel set  $A \in \mathcal{B}_\mathbb{R}$  and  $t \in [0, \infty)$ ,  $\mathbb{P}$ -a.s. where  $\mathbb{P}$  is the Wiener measure on  $C(\mathbb{R})$ , the space of continuous functions on  $\mathbb{R}$  (see [11]).

Note, that for every  $t \geq 0$ , the sequence  $l_n(t, x)$  defines a  $D$  valued process on  $X$ . Since,  $l(t, \cdot)$  is a.s. continuous on  $\mathbb{R}$ , we may also regard it as a  $D$  valued random variable.

In this paper, we prove that under the assumption that the Markov chain is strongly aperiodic (see definition 1), the weak invariance principle for  $W_n(t)$  implies an invariance principle for the local times, i.e. the convergence in distribution of  $W_n$  to  $W_\sigma$  implies that  $l_n(t, x)$  converges to  $l(t, x)$  (here, convergence is in the sense that there exists a probability space where  $l_n(t, x)$  converges to  $l(t, x)$  uniformly on compact sets). The case when  $X_i$ 's are i.i.d's was treated by Borodin [4], under the assumption that for every  $t \notin 2\pi\mathbb{Z}$ ,  $Ee^{itX_1} \neq 1$ , an assumption which we refer to as non-arithmeticity. We also show that the assumption of strong aperiodicity may be exchanged for the weaker

assumption of non-arithmeticity in the i.i.d case, and for a certain class of Markov chains (see the remark following Definition 1).

*Conventions and Notations:*

- $\mathbb{P}^x$  denotes the measure  $\mathbb{P}^\mu$ , where  $\mu$  assigns mass 1 to the point  $x \in S$ .
- $E^\mu$  denotes integrals on  $S$  with respect to the measure  $\mathbb{P}^\mu$  and by  $E^x$  integrals with respect to the measure  $\mathbb{P}^x$ .
- For two random variables  $X$  and  $Y$ ,  $X \stackrel{d}{=} Y$  means that  $X$  and  $Y$  are equally distributed. Also  $X_n \xrightarrow{d} X$  means that  $X_n$  converges in distribution to  $X$ .
- $\mathbb{C}^S$  denotes the space of complex valued functions on  $S$ , which is isomorphic to  $\mathbb{C}^n$  with  $n = |S|$ .
- $\mathcal{L}_S$  denotes the space of linear operators from  $\mathbb{C}^S$  into itself, and we denote by  $\|\cdot\|$  the operator norm on this space.
- Throughout the paper we assume that  $P$  is aperiodic and irreducible,  $\nu$  always denotes the stationary distribution with respect to  $P$ .

## 2 Statement of the Main Theorem.

Before stating the main theorem we introduce the notion of strong aperiodicity (see [6]):

**Definition 1** A function  $f : S \times S \rightarrow \mathbb{Z}$  is said to be aperiodic if the only solutions for

$$e^{itf(x,y)} = \lambda \frac{\varphi(x)}{\varphi(y)}, \quad \forall x, y \in S, \text{ such that } \mathbb{P}^x(y) > 0$$

with  $t \in \mathbb{R}$ ,  $|\lambda| = 1$ ,  $\varphi \in \mathbb{C}^S$ ,  $|\varphi| \equiv 1$  are  $t \in 2\pi\mathbb{Z}$ ,  $\varphi \equiv \text{const.}$   $f$  is said to be periodic if it is not aperiodic.

Since, we are only interested in the case  $f(x, y) = y$ , which gives  $f(X_{n-1}, X_n) = X_n$ , we say that the Markov Chain  $\{X_n\}$  is strongly aperiodic iff the function  $f(x, y) = y$  is aperiodic.

The assumption of strong aperiodicity may be dropped in case  $\{X_i\}$  is a sequence of i.i.d's and may be dropped in the Markov case provided that the underlying Markov shift is almost onto (see section 7). Note that strong aperiodicity of the Markov Chain implies aperiodicity of the stochastic matrix  $P$ , while the reverse implication is generally false - a simple random walk where  $X_n$  equals  $\pm 1$  with equal probability may serve as a counter example. Here for  $t = \pi$ ,  $\lambda = -1$  and  $\varphi \equiv 1$ ,

$$e^{i\pi X_n} = -1,$$

hence  $X_n$  is not strongly aperiodic. However

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is aperiodic. For discussion of condition 1 in a more general setting, see [2].

**Theorem 1** *Let  $\{X_n\}$  be an irreducible, strongly aperiodic, finite state Markov chain. Assume that  $E^\nu(X_0) = 0$  and*

$$W_n \xrightarrow{d} W_\sigma, \sigma > 0.$$

*Then there exists a probability space  $(X', \mathcal{B}', \mathbb{P}')$  and processes  $W'_n, W'_\sigma : X \rightarrow D_+$  such that:*

1.  $W_n \stackrel{d}{=} W'_n; W_\sigma \stackrel{d}{=} W'_\sigma$ .
2. *With probability one  $W'_n$  converges to  $W'_\sigma$  uniformly on compact subsets of  $[0, \infty)$ .*
3. *For every  $\epsilon, T > 0$  the processes  $l'_n(t, x)$  and  $l'(t, x)$  defined with respect to  $W'_n$  and  $W'_\sigma$  satisfy the relationship:*

$$\lim_{n \rightarrow \infty} \mathbb{P}' \left( \sup_{t \in [0, T], x \in \mathbb{R}} |l'_n(t, x) - l'(t, x)| > \epsilon \right) = 0. \quad (2.1)$$

We start by fixing the time variable at 1, and for convenience denote:  $t_n(x) \equiv l_n(1, x)$  and  $t(x) \equiv l(1, x)$ . The main step on the road to proving the previous theorem is to prove:

**Theorem 2** *Let  $\{X_n\}$  be a finite state Markov chain,  $E^\nu(X_0) = 0$  and*

$$W_n \xrightarrow{d} W_\sigma, \sigma > 0,$$

*then*

$$t_n \xrightarrow{d} t_\sigma.$$

In order to prove convergence in distribution of  $\{t_n(x)\}$ , we have to establish relative compactness of the family  $\Pi = \{t_n(x)\}_{n \in \mathbb{N}}$ , which by definition means that every sequence of elements from  $\Pi$  has a subsequence that converges in distribution. This is done in section 5. Then to complete the proof, in section 5.3 we show that  $t_\sigma(x)$  is the only possible distributional limit point.

*Organization of the paper:*

We begin with the proof under the assumption that  $X_n$  is strongly aperiodic. Section 3 contains estimates of the characteristic functions of the relevant processes, which are later used, in section 4 to carry out the calculations needed for the proof of Theorem 2. In sections 5 and 6 we prove Theorems 2 and 1 respectively under the assumption of strong aperiodicity.

In Section 7 we show the modifications needed to extend these results to the periodic case.

This work was motivated by Aaronson's work on *random walks in random sceneries* with independent jump random variables. In section 8 we explain the model of RWRS and why Theorem 1 implies Aaronson's result for RWRS with the distribution of a strongly aperiodic Markov Chain.

### 3 The Characteristic Function Operator $Q(t)$

From the irreducibility of  $P$  it follows that there is a unique probability distribution  $\nu$  on  $S$ , such that  $\nu P = \nu$ . This is equivalent to 1 being a simple eigenvalue of  $P$ , with a right-hand eigenspace of vectors in  $\mathbb{C}^S$  having equal entries.

Moreover, from aperiodicity of  $P$  it follows that all other eigenvalues of  $P$  are of modulus strictly less than 1. The projection to the eigenspace belonging to 1 is given by  $\langle v, \cdot \rangle \mathbf{1} = E^\nu(\cdot) \mathbf{1}$  (here,  $\mathbf{1}$  is a vector in  $\mathbb{C}^S$  with all entries equal to 1). In what follows, we omit the  $\mathbf{1}$ , so that depending on the context  $E^\nu(f)$  may denote both a scalar, or a vector in  $\mathbb{C}^S$  with all entries equal to  $E^\nu(f)$ . Hence, we can write

$$Pf = E^\nu(f) + Nf,$$

where  $N$  is an operator with spectral radius  $\rho(N)$  strictly less than 1. It follows that for every  $\rho(N) < \eta < 1$ , and  $n$  large enough, we have

$$\|N^n\| \leq \eta^n.$$

Let  $Q(t) : \mathbb{C} \rightarrow \mathcal{L}_S$  be an operator valued function defined by

$$Q(t)f(x) := \int_S e^{ity} f(y) \mathbb{P}^x(dy) = E^x(e^{itX_0} f) = \sum_{y \in S} P(x, y) e^{ity} f(y).$$

Note that  $Q(0) = P$ .

Since,  $S$  is a finite state space, it is easy to see that the function  $Q(t)$  is holomorphic and its derivatives are given by

$$\left( \frac{d^n}{dt^n} Q(t)f \right)(x) = i^n \int_S y^n e^{ity} f(y) \mathbb{P}^x(dy) = i^n \sum_{y \in S} y^n P(x, y) e^{ity} f(y).$$

Let

$$\varphi_n^\mu(t) = E^\mu(e^{itS_n}), \quad \varphi_n^\mu(t_1, t_2) = E^\mu(e^{it_1 S_n + it_2 X_n})$$

denote the characteristic functions of the processes  $S_n$  and  $(S_n, X_n)$  respectively, under the initial distribution  $\mu$ . Our primary interest in  $Q(t)$  is due to the fact that  $\varphi_n^\mu(t) = E^\mu(Q(t)^n \mathbf{1})$ , and  $\varphi_n^\mu(t_1, t_2) = E^\mu(Q(t_1)^n e^{it_2 S})$ .

### 3.1 Expansion of the largest eigenvalue.

It may be shown using standard perturbation techniques (see [9] or [7, Chapter 3]) that the largest eigenvalue of  $Q(t)$  and its eigenspaces are analytic functions of  $t$  in a neighborhood of 0. More precisely, we have that in a neighborhood  $I$  of 0,  $Q(t)$  has a simple eigenvalue  $\lambda(t)$  which is an analytic perturbation of  $\lambda(0)$ , and there is a positive gap between  $\lambda(t)$  and all other eigenvalues of  $Q(t)$ , i.e.

$$\inf_{t \in I} \min_{\tilde{\lambda}(t)} |\lambda(t) - \tilde{\lambda}(t)| \geq \text{const} > 0,$$

where the maximum is taken over all remaining eigenvalues of  $Q(t)$ . The projection function to the corresponding eigenspaces  $\Pi(t)$  is also analytic in  $t$ , and we may choose the eigenfunctions  $v(t)$  to be analytic perturbations of  $v(0) = \mathbf{1}$ .

Continuity of  $Q(t)$  immediately gives us that in a neighborhood  $I$  of 0

$$Q(t) = \lambda(t) \cdot \Pi(t) + N(t), \quad (3.1)$$

where  $\sup_{t \in I} \|N(t)\| = 1 - \eta < 1$ .

We proceed by evaluating the functions  $\lambda(t)$  and  $\Pi(t)$  for small  $t$ . By the structure of  $P = Q(0)$  discussed in the beginning of this section, we have

$$\begin{aligned} \lambda(0) &= 1, \\ v(0) &= 1, \\ \Pi(0)(\cdot) &= \langle \nu, \cdot \rangle v(0) = E^\nu(\cdot). \end{aligned} \quad (3.2)$$

**Lemma 1** *If  $E^\nu(X_0) = 0$ , then there exists a real nonnegative constant  $\sigma^2$  such that,*

$$\lambda(t) = 1 - \frac{\sigma^2}{2}t^2 + O(|t|^3),$$

and

$$\Pi(t)(\cdot) = E^\nu(\cdot) + O(|t|).$$

*Proof* The statement concerning  $\Pi(t)$  immediately follows from Taylor's expansion and formula (3.2), while the expansion of the main eigenvalue is carried out in [7, Lemma IV.4].

*Remark 1* It may be shown [7, Lemma IV.3, pp 24-25] that  $\sigma^2$  actually equals the asymptotic variance  $\lim_{n \rightarrow \infty} \frac{E^\nu(S_n^2)}{n}$  and therefore,  $\sigma^2 = 0$  corresponds to the case of a degenerate limit for  $W_n$ . Since this case is excluded in the main theorem, in what follows, we assume that the results of the previous lemma hold with  $\sigma^2 > 0$ .

### 3.2 The connection between strong aperiodicity and the spectrum of $Q(t)$ .

The next lemma connects the notion of strong aperiodicity with the spectrum of the characteristic function operator:

**Lemma 2** *An aperiodic and irreducible Markov chain  $\{X_n\}$  is strongly aperiodic iff  $\rho(Q(t)) < 1$  for all real  $t \neq 2\pi\mathbb{Z}$  ( $\rho(Q(t))$  is the spectral radius of  $Q(t)$ ).*

*Proof* Note that since  $Q(t)$  is a finite dimensional operator of norm less than or equal to 1,  $\rho(Q(t)) < 1$  is equivalent to demanding that  $Q(t)$  has no eigenvalues of modulus 1.

If  $\{X_n\}$  is not strongly aperiodic, we have

$$e^{ity} = \lambda \frac{\varphi(x)}{\varphi(y)}, \quad \forall x, y \in S \text{ such that } P^x(y) > 0$$

where  $|\lambda| = 1, |\varphi| \equiv 1, t \notin 2\pi\mathbb{Z}$  or  $t \in 2\pi\mathbb{Z}$  and  $\varphi \neq \text{const}$ . In the first case notice that,

$$(Q(t)\varphi)(x) = \int_S e^{ity} \varphi(y) \mathbb{P}^x(dy) = \lambda \varphi(x) \quad (3.3)$$

whence  $\rho(Q(t)) = 1$  for  $t \notin 2\pi\mathbb{Z}$ .

In the second case ( $t \in 2\pi\mathbb{Z}$ ), since  $y \in \mathbb{Z}$  we have

$$(Q(t)\varphi)(x) = (Q(0)\varphi)(x) = \int_S \varphi(y) P^x(dy) = \lambda \varphi(x).$$

This is impossible for non constant  $\varphi$  since we have a spectral gap property for  $t = 0$  (see the beginning of the section). This proves one direction of the iff statement.

To prove the opposite direction observe that if equation (3.3) holds with  $|\lambda| = 1$  we have

$$|\varphi(x)| = |(Q(t)\varphi)(x)| \leq \int_S |\varphi(y)| \mathbb{P}^x(dy) = (P|\varphi|)(x).$$

Since, 1 is a simple eigenvalue of  $P$ , we must have  $|\varphi| \equiv \text{const}$  and therefore, we may assume that  $|\varphi| \equiv 1$ . Hence since

$$\int_S e^{ity} \varphi(y) \mathbb{P}^x(dy) = \lambda \varphi(x),$$

and  $|\lambda \varphi(x)| = |\varphi(y)| = 1$ , it follows from Proposition 3 in the Appendix that

$$e^{ity} \varphi(y) = \lambda \varphi(x), \quad \forall x, y \in S, \text{ with } \mathbb{P}^x(y) > 0$$

and the claim follows.

#### 4 Estimates.

In this section we derive the main probability estimates needed to prove the main theorem. When lemma 1 is used, it is always assumed to hold with  $\xi > 0$  (see the remark that follows the lemma).

**Lemma 3** *If  $\{X_n\}$  is strongly aperiodic, then for every  $\delta > 0$ , there are  $N \in \mathbb{N}$ ,  $0 < r_\delta < 1$ ,  $c > 0$  such that for every  $n > N$ ,  $t \in [-\pi, \pi] \setminus (-\delta, \delta)$ ,*

$$\|Q(t)^n\| \leq cr_\delta^n.$$

*Proof* Fix  $\delta > 0$  and  $t \in [-\pi, \pi] \setminus (-\delta, \delta)$ . By strong aperiodicity (Lemma 2),  $r_t := r(Q(t)) < 1$ .

Fix,  $r_t < \tilde{r}_t < 1$  and choose  $n_0$  such that

$$r_t = \inf_{n \in \mathbb{N}} \|Q(t)^n\|^{\frac{1}{n}} \leq \|Q(t)^{n_0}\|^{\frac{1}{n_0}} < \tilde{r}_t.$$

By continuity of  $Q(\cdot)$  at  $t$ , there exists  $\eta = \eta_t > 0$ , such that if  $\tau \in (t - \eta, t + \eta)$ ,

$$\|Q(\tau)^{n_0}\| < \tilde{r}_t^{n_0}. \quad (4.1)$$

For  $n \in \mathbb{N}$ , write  $n = n_0 m + k$  where  $k < n_0$ . It follows from (4.1) that for every  $\tau \in (t - \eta, t + \eta)$ ,

$$\|Q(\tau)^n\| \leq \|Q(\tau)^{n_0}\|^m \cdot \|Q(\tau)^k\| \leq \tilde{r}_t^{n_0 m} \cdot \|Q(\tau)^k\| \leq \tilde{r}_t^n \cdot \frac{\|Q(\tau)\|^k}{\tilde{r}_t^k}.$$

Setting  $c_t = \sup \frac{\|Q(x)\|^k}{\tilde{r}_t^k}$  where the supremum is taken over all  $x \in (t - \eta, t + \eta)$ ,  $k \leq n_0$  we obtain

$$\|Q(\tau)^n\| \leq c_t \tilde{r}_t^n.$$

Hence, for every  $t \in [-\pi, \pi] \setminus (-\delta, \delta)$ , we can pick positive numbers  $n_t, \eta_t, c_t$  and  $r_t < 1$  such that for every  $n > n_t$  and  $\tau \in (t - \eta_t, t + \eta_t)$ ,

$$\|Q(\tau)^n\| \leq c_t r_t^n.$$

Since,  $[-\pi, \pi] \setminus (-\delta, \delta)$  is compact there exists a finite sequence  $\{t_i\}_{i=1, \dots, l}$  such that

$$[-\pi, \pi] \setminus (-\delta, \delta) \subset \bigcup_{i=1}^l (t_i - \eta_{t_i}, t_i + \eta_{t_i}).$$

The result follows by setting  $N = \max \{n_{t_i}\}$ ,  $r_\delta = \max \{r_{t_i}\}$ ,  $c = \max \{c_{t_i}\}$ .

**Lemma 4** *If  $\{X_n\}$  is strongly aperiodic and  $E^\nu(X_0) = 0$ , there exists a constant  $C$  such that for any initial distribution  $\mu$ ,  $n \in \mathbb{N}$ ,  $x \in \mathbb{Z}$ ,*

$$P^\mu(S_n = x) \leq \frac{C}{\sqrt{n}}.$$

*Proof* In view of equation (3.1) and lemma 1, there exist  $\delta, a, \eta > 0$  such that for  $t \in (-\delta, \delta)$ ,

$$Q(t) = \lambda(t) \Pi(t) + N(t)$$

where

$$|\lambda(t)| \leq 1 - at^2, \quad \|N(t)^n\| \leq (1 - \eta)^n.$$

By the inversion formula for Fourier transform,

$$\sqrt{n} \mathbb{P}^\mu(S_n = x) = \frac{\sqrt{n}}{2\pi} \int_{-\pi}^{\pi} E^\mu(Q(t)^n \mathbf{1}) e^{-itx} dt \quad (4.2)$$

$$\leq \left| \frac{\sqrt{n}}{2\pi} \int_{-\delta}^{\delta} E^\mu(Q(t)^n \mathbf{1}) e^{-itx} dt \right| + \frac{\sqrt{n}}{2\pi} \int_{R_\delta} \|Q(t)^n\| dt \quad (4.3)$$

By lemma 3, there are  $N \in \mathbb{N}$ ,  $0 < r_\delta < 1$ ,  $c > 0$  such that for any  $n > N$ ,

$$\frac{\sqrt{n}}{2\pi} \int_{R_\delta} \|Q(t)^n\| dt \leq c\sqrt{n}r_\delta^n.$$

Therefore, the second term in (4.3) is uniformly bounded in  $n$ . We proceed with estimating the first term in (4.3).

$$\left| \frac{\sqrt{n}}{2\pi} \int_{-\delta}^{\delta} E^\mu (Q(t)^n \mathbf{1}) e^{-itx} dt \right| \leq \frac{\sqrt{n}}{2\pi} \int_{-\delta}^{\delta} |\lambda(t)|^n dt + \frac{\sqrt{n}}{2\pi} \int_{-\delta}^{\delta} \|N(t)^n\| dt \quad (4.4)$$

$$\leq \frac{\sqrt{n}}{2\pi} \int_{-\delta}^{\delta} (1 - at^2)^n dt + \frac{2\delta\sqrt{n}}{2\pi} (1 - \eta)^n. \quad (4.5)$$

It is obvious that the second term in (4.5) tends to 0. To see that the first term is bounded, a change of variables  $t = \frac{x}{\sqrt{n}}$  gives us,

$$\begin{aligned} \frac{\sqrt{n}}{2\pi} \int_{-\delta}^{\delta} (1 - at^2)^n dt &= \frac{1}{2\pi} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \left(1 - \frac{ax^2}{n}\right)^n dx \\ &\leq \frac{1}{2\pi} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-ax^2} dx. \end{aligned}$$

Since, the last integral is uniformly bounded in  $n$ , this completes the proof.

**Lemma 5** (*Estimation of the Potential Kernel, see [8]*) *If  $\{X_n\}$  is strongly aperiodic and  $E^\nu(X_0) = 0$ , then there exists  $C > 0$  such that for every  $x, y \in \mathbb{Z}$  and  $s \in S$ , and initial distribution  $\mu$ ,*

$$\sum_{n=0}^{\infty} |\mathbb{P}^\mu(S_n = x, X_n = s) - \mathbb{P}^\mu(S_n = y, X_n = s)| \leq C|x - y|.$$

*Proof* We denote by  $\gamma_t(\cdot) : S \rightarrow \mathbb{T}$  the function  $e^{it(\cdot)}$  from the state space  $S$  to the circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .

For simplicity of notation we assume that  $y = 0$ . The generalization to the case when  $y \neq 0$  is straightforward. Note that since  $x \in \mathbb{Z}$ , its enough to prove that the term converges for every  $x$  and is  $O(|x|)$  as  $|x| \rightarrow \infty$ .

By lemma 1 and equation (3.1) we may fix  $\delta > 0$  and positive constants  $C_1, C_2, C_3, \eta$  such that for  $t \in (-\delta, \delta)$ ,

$$\begin{aligned} |\lambda(t)| &\leq 1 - C_1 t^2 \\ |\Im \lambda(t)| &\leq C_2 |t^3| \\ \|N(t)\| &\leq 1 - \eta \\ \Pi(t)(\cdot) &= E^\nu(\cdot) + \epsilon(t)(\cdot) \end{aligned}$$

where  $\|\epsilon(t)\| \leq C_3 \cdot |t|$ . Set  $R_\delta = [-\pi, \pi] \setminus (-\delta, \delta)$ .

By the inversion formula for the Fourier Transform,

$$\sum_{n=0}^{\infty} \left| \mathbb{P}^\mu(S_n = 0, X_n = s) - \mathbb{P}^\mu(S_n = x, X_n = s) \right| \quad (4.6)$$

$$= \sum_{n=0}^{\infty} \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} E^\mu(Q(t_1)^n \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \quad (4.7)$$

By lemma 3 there are positive constants  $n_0, c, r_\delta < 1$  such that for every  $n \geq n_0, t \in R_\delta$ ,

$$\|Q(t)^n\| \leq cr_\delta^n.$$

Thus,

$$\begin{aligned} & \sum_{n=n_0}^{\infty} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{R_\delta} \left| E^\mu(Q(t_1)^n \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} \right| dt_1 dt_2 \\ & \leq \sum_{n=n_0}^{\infty} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{R_\delta} cr_\delta^n \cdot 2 dt_1 dt_2 < \infty \end{aligned}$$

Here  $R_\delta = [-\pi, \pi] \setminus (-\delta, \delta)$ . It follows that

$$\sum_{n=0}^{\infty} \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{R_\delta} E^\mu(Q(t_1)^n \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| < \infty$$

which is enough for the statement of the lemma. Hence, it remains to estimate expression (4.7) where the inner integration is carried over the set  $(-\delta, \delta)$ .

We have,

$$\begin{aligned} & \sum_{n=0}^{\infty} \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} E^\mu(Q(t_1)^n \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \\ & \leq \sum_{n=0}^{\infty} \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} E^\mu(\lambda(t_1)^n \Pi(t_1) \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \\ & + \sum_{n=0}^{\infty} \left| \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} E^\mu(N(t_1)^n \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right| \end{aligned} \quad (4.8)$$

Since,  $\|N(t)\| \leq 1 - \eta < 1$  and  $|\gamma_t(\cdot)| \equiv 1$ , we have

$$\sum_{n=0}^{\infty} \left| \int_{-\pi-\delta}^{\pi} \int_{-\delta}^{\delta} \frac{1}{(2\pi)^2} E^{\mu} (N(t_1)^n \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right| \leq \sum_{n=0}^{\infty} C' \cdot (1 - \eta)^n \leq C' \frac{1}{\eta} \quad (4.9)$$

where  $C'$  is some constant, which is again enough for our purposes.

We proceed with estimating (4.8):

$$\begin{aligned} & \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi-\delta}^{\pi} \int_{-\delta}^{\delta} E^{\mu} (\lambda(t_1)^n \Pi(t_1) \gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \\ & \leq \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi-\delta}^{\pi} \int_{-\delta}^{\delta} \lambda(t_1)^n E^{\nu} (\gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \\ & + \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi-\delta}^{\pi} \int_{-\delta}^{\delta} E^{\mu} (\lambda(t_1)^n \cdot \epsilon(t_1)(\gamma_{t_2})) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right|. \end{aligned} \quad (4.10)$$

Since,  $E^{\nu}(\gamma_{t_2}) = E^{\nu}(e^{it_2 X_0})$ , by using the inversion formula and the Fubini theorem we obtain,

$$\begin{aligned} & \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi-\delta}^{\pi} \int_{-\delta}^{\delta} \lambda(t_1)^n E^{\nu} (\gamma_{t_2}) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \\ & = \left| \Re \left\{ \frac{\nu_s}{2\pi} \int_{-\delta}^{\delta} \lambda(t)^n (1 - e^{-itx}) dt \right\} \right| \leq \left| \Re \left\{ \frac{1}{2\pi} \int_{-\delta}^{\delta} \lambda(t)^n \cdot (1 - e^{-itx}) dt \right\} \right| \\ & \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} |\lambda(t)|^n \cdot (1 - \cos tx) dt + \frac{1}{2\pi} \int_{-\delta}^{\delta} |\Im \{\lambda(t)^n\}| |\sin tx| dt. \end{aligned} \quad (4.11)$$

By summing over  $n$  the first term in expression (4.11) and by our choice of  $\delta$ ,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\delta}^{\delta} |\lambda(t)|^n \cdot (1 - \cos tx) dt & \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} \sum_{n=0}^{\infty} (1 - C_1 t^2)^n (1 - \cos tx) dt \\ & = \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{1}{C_1 t^2} (1 - \cos tx) dt \end{aligned}$$

Now,

$$\frac{1}{2\pi} \int_{-\frac{1}{x}}^{\frac{1}{x}} \frac{1}{C_1 t^2} (1 - \cos tx) dt = O(|x|), (|x| \rightarrow \infty)$$

and

$$\frac{1}{2\pi} \int_{\frac{1}{x}}^{\delta} \frac{1}{C_1 t^2} (1 - \cos tx) dt \leq \frac{1}{\pi} \int_{\frac{1}{x}}^{\delta} \frac{1}{C_1 t^2} dt = O(|x|), (|x| \rightarrow \infty).$$

Hence, from the integrand being an even function, we have

$$\sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\delta}^{\delta} |\lambda(t)|^n \cdot (1 - \cos tx) dt = O(|x|), (|x| \rightarrow \infty) \quad (4.12)$$

To estimate the sum over  $n$  of the second term in expression (4.11) we note that,

$$|\Im \{ \lambda(t)^n \}| \leq |\Im \lambda(t)| \cdot n |\lambda(t)|^{n-1} \leq C_2 t^3 \cdot n (1 - C_1 t^2)^{n-1}$$

where the first inequality is easily seen to be true by induction on  $n$ . Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\delta}^{\delta} |\Im \{ \lambda(t)^n \}| |\sin tx| dt &\leq \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{C_2 |t|^3}{C_1 t^4} |\sin tx| dt \\ &= O(|x|), (|x| \rightarrow \infty) \end{aligned} \quad (4.13)$$

This completes the estimation of the sum over  $n$  of the first term in expression (4.10) and all is left is to estimate the sum over  $n$  of the second term in expression (4.10):

$$\begin{aligned} &\sum_{n=0}^{\infty} \left| \Re \left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} E^{\mu}(\lambda(t_1)^n \epsilon_{t_1}(\gamma_{t_2})) \cdot (1 - e^{-it_1 x}) \cdot e^{-it_2 s} dt_1 dt_2 \right\} \right| \\ &\leq \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} \sum_{n=0}^{\infty} (1 - C_1 t_1^2)^n C_3 |t_1| \cdot |1 - e^{-it_1 x}| dt_1 dt_2 \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} \frac{C_3}{C_1 |t_1|} \cdot |1 - e^{-it_1 x}| dt_1 dt_2 = O(|x|), (|x| \rightarrow \infty) \end{aligned} \quad (4.14)$$

The lemma follows by combining results (4.9), (4.12), (4.13) and (4.14).

## 5 Proof of theorem 2.

Throughout this section we assume that  $\{X_n\}$  satisfies the assumptions of Theorem 2.

5.1 A fourth moment inequality for  $L_n(x) - L_n(y)$ .

**Lemma 6** *There exists  $C > 0$  such that for every  $x, y \in \mathbb{Z}$ ,*

$$E^\mu \left( (L_n(x) - L_n(y))^4 \right) \leq C \cdot n \cdot |x - y|^2.$$

*Proof* Set  $Z_n = (S_n, X_n)$ . Throughout the proof we use the fact that for any initial distribution  $\mu$  on  $S$ ,  $x, y \in \mathbb{Z}$ ,  $n, j \in \mathbb{N}$ , ( $j \leq n$ ) and  $s_1, s_2 \in S$ ,

$$\mathbb{P}^\mu (Z_n = (x, s_2) | Z_j = (y, s_1)) = \mathbb{P}^{s_1} (Z_{n-j} = (x - y, s_2)). \quad (5.1)$$

Let  $\mathbf{1}_z(t)$  be the indicator function of the set  $\{z\}$  and set

$$\psi(t) = \mathbf{1}_x(t) - \mathbf{1}_y(t).$$

We have

$$E^\mu \left( (L_n(x) - L_n(y))^4 \right) = \sum_{\bar{i}} E^\mu \left( \prod_{j=1}^4 \psi(S_{i_j}) \right)$$

where the summation is carried over all vectors  $\bar{i} = (i_1, i_2, i_3, i_4)$  with coordinates equal to integers between 0 and  $n$ .

By symmetry, we may assume that  $i_1 \leq i_2 \leq i_3 \leq i_4$ . For a fixed  $\bar{i}$

$$E^\mu \left( \prod_{j=1}^4 \psi(S_{i_j}) \right) = \sum_{\xi} (-1)^\sigma E^\mu (\mathbf{1}_{\xi_1}(S_{i_1}) \psi(S_{i_2}) \mathbf{1}_{\xi_2}(S_{i_3}) \psi(S_{i_4})) \quad (5.2)$$

where the sum is carried out over all vectors  $\bar{\xi} = (\xi_1, \xi_2)$  with coordinates equal to either  $x$  or  $y$  and  $\sigma = \sigma(\bar{\xi})$  is the number of coordinates that equal  $y$ .

Fixing  $\bar{i}$  and  $\bar{\xi}$ , and using (5.1) we obtain

$$E^\mu (\mathbf{1}_{\xi_1}(S_{i_1}) \psi(S_{i_2}) \mathbf{1}_{\xi_2}(S_{i_3}) \psi(S_{i_4})) = \sum_{s \in S^4} [\mathbb{P}^\mu (Z_{i_1} = (\xi_1, s_1)) \cdot F(i_1, i_2, i_3) \cdot (\mathbb{P}^{s_3}(S_{i_4} = x) - \mathbb{P}^{s_3}(S_{i_4} = y))] \quad (5.3)$$

where

$$F(i_1, i_2, i_3) = [\mathbb{P}^{s_1} \{Z_{i_2-i_1} = (x - \xi_1, s_2)\} \mathbb{P}^{s_2} \{Z_{i_3-i_2} = (\xi_2 - x, s_3)\} \\ - \mathbb{P}^{s_1} \{Z_{i_2-i_1} = (y - \xi_1, s_2)\} \mathbb{P}^{s_2} \{Z_{i_3-i_2} = (\xi_2 - y, s_3)\}].$$

At this point we take absolute values in (5.3) and sum over all vectors  $\bar{i}$  obtaining

$$\left| \sum_{\bar{i}} E^\mu (\mathbf{1}_{\xi_1}(S_{i_1}) \psi(S_{i_2}) \mathbf{1}_{\xi_2}(S_{i_3}) \psi(S_{i_4})) \right| \leq \sum_{\bar{s} \in S^4} \sum_{\bar{i}} |\mathbb{P}^\mu(Z_{i_1} = (\xi_1, s_1)) \cdot \\ F(i_1, i_2, i_3) \cdot (\mathbb{P}^{s_3}(S_{i_4} = x) - \mathbb{P}^{s_3}(S_{i_4} = y))| \\ \leq \sum_{\bar{s} \in S^4} \sum_{0 \leq i_1 \leq i_2 \leq i_3 \leq n} \left[ C \cdot \frac{1}{\sqrt{i_1 + 1}} \cdot |F(i_1, i_2, i_3)| \cdot |x - y| \right]$$

where  $C$  is some constant. Here, Lemma 4 and Lemma 5 were used to obtain the last inequality and  $i_1 + 1$  appears instead of  $i_1$ , so that the expression will be defined for  $i_1 = 0$ .

Note that,

$$|F(i_1, i_2, i_3)| \leq \mathbb{P}^{s_1}(Z_{i_2-i_1} = (x - \xi_1, s_2)) |\mathbb{P}^{s_2}\{Z_{i_3-i_2} = (\xi_2 - x, s_3)\} - \mathbb{P}^{s_2}\{Z_{i_3-i_2} = (\xi_2 - y, s_3)\}| \\ + \mathbb{P}^{s_2}\{Z_{i_3-i_2} = (\xi_2 - y, s_3)\} |\mathbb{P}^{s_1}\{Z_{i_2-i_1} = (x - \xi_1, s_2)\} - \mathbb{P}^{s_1}\{Z_{i_2-i_1} = (y - \xi_1, s_2)\}|$$

Substituting  $|F(i_1, i_2, i_3)|$  by the above expression, using Lemma 4 to deduce that sums of the type

$$\sum_{k=0}^n P^\mu(Z_n = (x, s))$$

are bounded by constant times  $\sqrt{n}$ , and applying Lemma 5 we conclude that

$$\left| \sum_{\bar{i}} E(\mathbf{1}_{\xi_1}(S_{i_1}) \psi(S_{i_2}) \mathbf{1}_{\xi_2}(S_{i_3}) \psi(S_{i_4})) \right| \leq C \cdot n |x - y|^2.$$

Since there is only a finite number of possible vectors  $\xi$  in (5.2) (there are exactly 4) the claim follows.

**Corollary 1** For every  $x, y \in \mathbb{R}$

$$\mathbb{P}^\mu(|t_n(x) - t_n(y)| > \epsilon) \leq \frac{C}{\epsilon^4} |x - y|^2.$$

*Proof* By Lemma 6,

$$E^\mu((t_n(x) - t_n(y))^4) = \frac{1}{n^2} E^\mu[(L_n(\lfloor \sqrt{n}x \rfloor) - L_n(\lfloor \sqrt{n}y \rfloor))^4] \\ \leq C |x - y|^2.$$

The claim now follows by Chebychev's inequality.

5.2 Relative compactness of  $t_n(x)$  in  $D$ .

A sequence  $\{X_n\}$  of random variables taking values in a standard Borel Space  $(X, \mathcal{B})$  is called tight if for every  $\epsilon > 0$  there exists a compact  $K \subset X$  such that for every  $n \in \mathbb{N}$ ,

$$P_n(K) > 1 - \epsilon,$$

where  $P_n$  denotes the distribution of  $X_n$ . By Prokhorov's Theorem relative compactness of  $t_n(x)$  in  $D$  is equivalent to tightness. Therefore we are interested in characterizing tightness in  $D$ .

For  $x(t)$  in  $D_{[-m, m]}, T \subseteq [-m, m]$  set

$$\omega_x(T) = \sup_{s, t \in T} |x(s) - x(t)|$$

and

$$\omega_x(\delta) := \sup_{|s-t| < \delta} |x(s) - x(t)|.$$

$\omega_x(\delta)$  is called the modulus of continuity of  $x$ . Due to the Arzela - Ascoli theorem it plays a central role in characterizing tightness in the space  $C[-m, m]$  of continuous functions on  $[-m, m]$ , with a Borel  $\sigma$ -algebra generated by the topology of uniform convergence.

The function that plays in  $D[-m, m]$  the role that the modulus of continuity plays in  $C[-m, m]$  is defined by

$$\omega'_x(\delta) = \inf_{\{t_i\}} \max_{1 \leq i \leq v} \omega([t_i, t_{i+1})),$$

where  $\{t_i\}$  denotes a  $\delta$  sparse partition of  $[-m, m]$ , i.e.  $\{t_i\}$  is a partition  $-m = t_1 < t_2 < \dots < t_{v+1} = m$  such that  $\min_{1 \leq i \leq v} |t_{i+1} - t_i| > \delta$ . It is easy to check that if  $\frac{1}{2} > \delta > 0$ , and  $m \geq 1$ ,

$$\omega'_x(\delta) \leq \omega_x(2\delta).$$

For details see [3, Sections 12 and 13]. The next theorem is a characterization of tightness in the space  $D$ .

**Theorem 3** [3, Lemma 3, p.173] (1) The sequence  $t_n$  is tight in  $D$  if and only if its restriction to  $[-m, m]$  is tight in  $D_{[-m, m]}$  for every  $m \in \mathbb{R}_+$ .

(2) The sequence  $t_n$  is tight in  $D_{[-m, m]}$  if and only if the following two conditions hold:

$$(i) \forall x \in [-m, m], \lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} P[|t_n(x)| \geq a] = 0.$$

$$(ii) \forall \epsilon > 0, \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P[\omega'_{t_n}(\delta) \geq \epsilon] = 0.$$

*Remark 2* See [3, Thm. 13.2] and Corollary after wards. Conditions (i) and (ii) of the previous theorem imply that

$$\lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left[ \sup_{x \in [-m, m]} |t_n(x)| \geq a \right] = 0.$$

**Proposition 1** *The sequence  $\{t_n(x)\}_{n=1}^\infty$  is tight.*

*Proof* We prove that condition 2(i) holds.

Fix  $\epsilon > 0$ ,  $x \in \mathbb{R}$ . Since the Brownian Motion  $W_\sigma(t)$  satisfies

$$\lim_{M \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0,1]} |W_\sigma(t)| > M \right) = 0$$

and  $W_n(t)$  converges in distribution to  $W_\sigma(t)$  there are  $M, n_0$  such that for all  $n > n_0$ ,

$$\mathbb{P}^\mu \left( \sup_{t \in [0,1]} |W_n(t)| > M \right) < \epsilon.$$

By definition of  $t_n(x)$ , it follows that if  $|x| > M$ ,  $n > n_0$ ,

$$\mathbb{P}^\mu (|t_n(x)| > 0) < \epsilon.$$

Now, if  $|x| \leq M$ , by corollary 1,

$$\begin{aligned} \mathbb{P}^\mu (|t_n(x)| > a) &\leq \mathbb{P}^\mu (|t_n(M+1)| > 0) + \mathbb{P}^\mu (|t_n(x) - t_n(M+1)| > a) \\ &\leq \epsilon + \frac{4C \cdot (M+1)^2}{a^4} \end{aligned}$$

and the last expression can be made less than  $2\epsilon$  for sufficiently large  $a$ .

To prove condition 2(ii) WLOG we may assume that  $m \geq 1$ . Since  $\omega'_x(\delta) \leq \omega_x(2\delta)$ , it is sufficient to prove that the stronger condition

$$\forall \epsilon > 0. \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^\mu \left[ \sup_{x, y \in [-m, m]; |x-y| < \delta} |t_n(x) - t_n(y)| \geq \epsilon \right] = 0 \quad (5.4)$$

holds.

Let  $\epsilon > 0$ . By Corollary 1 there exists  $C > 0$  such that for all  $x, y : \frac{1}{\sqrt{n}} \leq |x-y| \leq 1$

$$\mathbb{P}^\mu (|t_n(x) - t_n(y)| > \epsilon) \leq \frac{C}{\epsilon^4} |x-y|^2. \quad (5.5)$$

Let  $\delta > 0$  and  $n > \delta^{-2}$ , notice that  $t_n(x)$  is constant on segments of the form  $\left[ \frac{j}{\sqrt{n}}, \frac{j+1}{\sqrt{n}} \right)$ , hence

$$\mathbb{P}^\mu \left( \sup_{x, y \in [-m, m]; |x-y| < \delta} |t_n(x) - t_n(y)| \geq 4\epsilon \right) \leq \sum_{|k\delta| \leq m} \mathbb{P}^\mu \left( \sup_{k\delta\sqrt{n} \leq j \leq (k+1)\delta\sqrt{n}} \left| t_n(k\delta) - t_n\left(\frac{j}{\sqrt{n}}\right) \right| \geq \epsilon \right).$$

By [3, Theorem 10.2] it follows from (5.5) that there exists  $C_2 > 0$  such that

$$\mathbb{P}^\mu \left( \sup_{k\delta\sqrt{n} \leq j \leq (k+1)\delta\sqrt{n}} \left| t_n(k\delta) - t_n\left(\frac{j}{\sqrt{n}}\right) \right| \geq \epsilon \right) \leq \frac{C_2}{\epsilon^4} \delta^2. \quad (5.6)$$

Therefore,

$$\mathbb{P}^\mu \left( \sup_{x, y \in [-m, m]; |x-y| < \delta} |t_n(x) - t_n(y)| \geq 4\epsilon \right) \leq \frac{2C_2 m}{\epsilon^4} \delta \xrightarrow{\delta \rightarrow 0} 0.$$

5.3 Identifying  $t_\sigma(x)$  as the Only Possible Limit of a Subsequence of  $\{t_n(x)\}_{n \in \mathbb{N}}$ .

**Proposition 2** *Assume that the sequence  $\{X_n\}$  satisfies the assumptions of Theorem 2. Let  $t_{n_k}(x)$  be some subsequence of  $t_n(x)$  that converges in distribution to some limit  $q(x)$ . Then  $q(x) \stackrel{d}{=} t_\sigma(x)$ .*

*Proof* Let  $q$  be a limit point of  $\{t_n(x)\}$ . i.e. there exists  $n_k \rightarrow \infty$  such that  $t_{n_k} \xrightarrow{d} q(x)$ .

By [10, Section 2], for every  $a, b \in \mathbb{R}$ , the function on  $D_{[0,1]}$  defined by

$$f \rightarrow \int_0^1 \mathbf{1}_{[a,b]}(f) dt$$

is continuous in the Skorokhod topology.

Therefore, since  $W_n(t) \xrightarrow{d} W_\sigma(t)$ , for every  $a, b$  the random variables

$$A_n([a, b]) = \int_0^1 \mathbf{1}_{[a,b]}(W_n(t)) dt$$

converge in distribution to the occupation measure of the Brownian motion defined by

$$A([a, b]) = \int_0^1 \mathbf{1}_{[a,b]}(W_\sigma(t)) dt.$$

The local time  $t_\sigma$  of the Brownian motion is an almost surely continuous function of  $x$  such that for fixed  $a, b \in \mathbb{R}$  the equality

$$\int_a^b t_\sigma(x) dx = \int_0^1 \mathbf{1}_{[a,b]}(W_\sigma(t)) dt \quad (5.7)$$

holds almost surely (see [11]).

By straightforward calculations using definitions, we have

$$\left| \int_0^1 \mathbf{1}_{[a,b]}(W_n(t)) dt - \int_a^b t_n(x) dx \right| \leq \int_{\frac{\lfloor \sqrt{n}a \rfloor}{\sqrt{n}}}^{\frac{\lfloor \sqrt{n}a \rfloor + 1}{\sqrt{n}}} t_n(x) dx + \int_{\frac{\lfloor \sqrt{n}b \rfloor}{\sqrt{n}}}^{\frac{\lfloor \sqrt{n}b \rfloor + 1}{\sqrt{n}}} t_n(x) dx. \quad (5.8)$$

Now,

$$\begin{aligned} \mathbb{P}^\mu \left( \left| \int_{\frac{|\sqrt{na}|}{\sqrt{n}}}^{\frac{|\sqrt{na}|+1}{\sqrt{n}}} t_n(x) dx \right| > \epsilon \right) &\leq \mathbb{P}^\mu \left( \sup_{x \in [a-1, b+1]} |t_n(x)| > M \right) \\ &+ \mathbb{P}^\mu \left( \left| \int_{\frac{|\sqrt{na}|}{\sqrt{n}}}^{\frac{|\sqrt{na}|+1}{\sqrt{n}}} t_n(x) dx \right| > \epsilon, \sup_{x \in [a-1, b+1]} |t_n(x)| \leq M \right). \end{aligned}$$

The second summand on the right side of the above inequality tends to 0 since the integral is less than  $\frac{M}{\sqrt{n}}$ . The first summand is arbitrarily close to 0 for  $M, n$  large enough, by Remark 2. Same reasoning applied to both summands of equation (5.8) gives

$$\left| \int_0^1 \mathbf{1}_{[a,b)}(W_n(t)) dt - \int_a^b t_n(x) dx \right| \xrightarrow{d} 0.$$

It follows that the distributional limits of  $\int_0^1 \mathbf{1}_{[a,b)}(W_n(t)) dt$  and  $\int_a^b t_n(x) dx$  coincide (see [3, Thm.3.1]). Therefore, for every  $a, b \in \mathbb{R}$ ,

$$\int_a^b t_n(x) dx \xrightarrow{d} \int_a^b t_\sigma(x) dx.$$

For every  $a, b$  the function

$$f \mapsto g_{a,b}(f) = \int_a^b f(x) dx$$

is a continuous function from  $D$  to  $\mathbb{R}$ . Therefore, since  $t_{n_k} \xrightarrow{d} q(x)$ , it follows that

$$\int_a^b t_{n_k}(x) dx \xrightarrow{d} \int_a^b q(x) dx.$$

Hence,

$$\int_a^b q(x) dx = \int_a^b t_\sigma(x) dx.$$

Since the collection of functions  $g_{a,b}, a, b \in \mathbb{R}$  generates  $\mathcal{G}(D)$ , it follows that

$$q \stackrel{d}{=} t_\sigma.$$

## 6 Proof of Borodin's Theorem (Theorem 1).

We start by forming the mutual probability space on which,  $W'_n$  and  $W'_\sigma$  are defined.

Let  $\{t_i\}$  be a dense subset of  $[0, \infty]$  and for each  $n$  let us regard the infinite vector

$$\xi_n := (W_n, l_n(t_1, \cdot), l_n(t_2, \cdot), \dots)$$

as a random variable taking values in the space  $\Pi := D_+ \times D^\mathbb{N}$

Since, a countable product of Polish Spaces is Polish, to prove convergence of the sequence  $\{\xi_n\}$  we need to establish tightness and uniquely identify the limit. By Tichonov's theorem, tightness in each coordinate of the vectors  $\xi_n$  separately (established in Theorem 2) implies tightness for the whole sequence  $\{\xi_n\}$ .

To identify the limit, generalizing the method in the proof of proposition 2 to  $\Pi$ , for each sequence  $\alpha = \{(a_i, b_i)\}_{i \in \mathbb{N}}$  we define a continuous function  $g_\alpha$  on  $\Pi$  (taking values in  $D_+ \times \mathbb{R}^\mathbb{N}$ ) by

$$(f, h_1, h_2, \dots) \xrightarrow{g_\alpha} \left( f, \int_{a_1}^{b_1} h_1, \int_{a_2}^{b_2} h_2, \dots \right).$$

The functions  $\{g_\alpha\}$  where  $\alpha$  goes over all possible sequences of intervals in  $\mathbb{R}^2$  generate the Borel  $\sigma$ -field of  $\Pi$ . Now, the arguments in the proof of proposition (2) may be easily modified to show that

$$g_\alpha(\xi_n) \xrightarrow{d} g_\alpha(\xi)$$

where  $\xi := (W'_\sigma, l(t_1, \cdot), l(t_2, \cdot), \dots)$ . This uniquely identifies the limit, establishing the convergence in distribution of the sequence  $\xi_n$  to  $\xi$ . Now, by Skorokhod's theorem (See [3, Theorem 6.7]), we may define processes  $\xi'_n, \xi' : \Omega' \rightarrow$  on some probability space  $(\Omega', \mathcal{B}', \mathbb{P})$  such that  $\xi'_n$  and  $\xi'$  have the same distribution as  $\xi_n$  and  $\xi$  respectively, and such that almost surely,  $\xi'_n$  converges to  $\xi'$ . Note that for  $n \in \mathbb{N}$ , by left continuity, for almost every  $\omega \in \Omega'$ , the vector

$$\xi'_n = (W'_\sigma, l'_n(t_1, x), l'_n(t_2, x), \dots)$$

uniquely determines a function  $l'_n[t, x] : [0, 1] \times (-\infty, \infty) \rightarrow \mathbb{R}$ , which coincides with the coordinates of  $\xi'_n$  for  $t \in \{t_i\}$ , and is Cadlag in each variable.

It is clear that the functions  $l'_n(t, x)$  are a.s equal to the local time of  $W'_n$ . Similarly, the vector  $\xi'$  uniquely, determines a function  $l'(t, x)$  which a.s equals the Brownian local time of  $W'_\sigma$ , and is therefore continuous. The a.s. convergence of  $\xi'_n$  to  $\xi'$  implies a.s. convergence in the  $J_1$  topology of  $W'_n$  to  $W'_\sigma$ , thus proving (2) of theorem 1.

It remains to prove that  $l'_n, l', n = 1, 2, \dots$  satisfy the statement in (3). To see this fix  $T > 0$ . For  $h > 0$ , whose value is to be determined later, we extract from  $\{t_i\}$  a finite partition of  $[0, T]$  with mesh less than  $h$  and denote it by

$$T : 0 = t_0 < t_1 < \dots < t_l = 1.$$

Now for  $i = 1, \dots, l$ , the process  $l'_n(t_i, \cdot)$ , a.s converges to  $l'(t_i, \cdot)$  in the metric of the space  $D$ . Since, the limit is a continuous function, by properties of the  $J_1$  topology, this convergence must be uniform on compact subsets of  $\mathbb{R}$ . We may conclude that for every compact subset  $K \subseteq \mathbb{R}$ , every  $\epsilon > 0$ , and almost all  $\omega \in \Omega'$ , there exists  $N_0$ , such

that for every  $n > N_0$ ,

$$\sup_{x \in K, t \in T} |l'_n(t, x) - l'(t, x)| < \epsilon. \quad (6.1)$$

Now, by monotonicity of the local time as a function of time, if  $l'_n(t, x) - l'(t, x) \geq 0$  then

$$l'_n(t, x) - l'(t, x) \leq l'_n(t_{i+1}, x) - l'(t_i, x)$$

and if  $l'_n(t, x) - l'(t, x) \leq 0$  then

$$l'(t, x) - l'_n(t, x) \leq l'(t_{i+1}, x) - l'_n(t_i, x)$$

where  $t_i, t_{i+1}$  are points in  $T$  satisfying

$$t_i \leq t \leq t_{i+1}.$$

Hence,

$$\mathbb{P} \left( \sup_{t \in [0, 1], x \in \mathbb{R}} |l'_n(t, x) - l'(t, x)| > 2\epsilon \right) \leq \mathbb{P} \left( \sup_{t \in [0, 1]} |W'_n| > M \right) + \mathbb{P} \left( \sup_{t \in [0, 1]} |W'_\sigma| > M \right) \quad (6.2)$$

$$+ \mathbb{P} \left( \sup_{i, x \in [-M, M]} |l'_n(t_{i+1}, x) - l'(t_i, x)| > \epsilon \right) \quad (6.3)$$

$$+ \mathbb{P} \left( \sup_{i, x \in [-M, M]} |l'_n(t_i, x) - l'(t_{i+1}, x)| > \epsilon \right) \quad (6.4)$$

where  $i = 1, \dots, l$ .

As in the proof of Proposition (1) expression (6.2) is small for sufficiently large  $M$ . Expressions (6.3) and (6.4) are handled similarly. We have

$$\begin{aligned} \mathbb{P} \left( \sup_{i, x \in [-M, M]} |l'_n(t_{i+1}, x) - l'(t_i, x)| > \epsilon \right) &\leq \mathbb{P} \left( \sup_{t \in T, x \in [-M, M]} |l'_n(t_{i+1}, x) - l'(t_{i+1}, x)| > \frac{\epsilon}{2} \right) \\ &\quad + \mathbb{P} \left( \sup_{i=0, 1, \dots, l, x} |l'(t_{i+1}, x) - l'(t_i, x)| > \frac{\epsilon}{2} \right). \end{aligned}$$

The first expression on the right equals 0 for all  $n$  sufficiently large, by 6.1. The second expression, by a.s. continuity of Brownian local time, can be made arbitrarily small by taking  $h$  to be sufficiently small. This concludes the proof.

## 7 The periodic Case

Let  $X_1, X_2, \dots$  be an i.i.d. sequence in the domain of attraction of the Gaussian distribution. In this setting Borodin proved convergence of local times under the condition of what we call non-arithmeticity of the random walk  $S_n$ , that

is

$$E\left(e^{itX_i}\right) = 1 \Leftrightarrow t \in 2\pi\mathbb{Z}.$$

Our notion of strong aperiodicity in the i.i.d case is equivalent to the stronger condition that

$$\left|E\left(e^{itX_i}\right)\right| = 1 \Leftrightarrow t \in 2\pi\mathbb{Z}.$$

In this section we show that the methods of this paper are sufficient to replace the assumption of strong aperiodicity by a weaker non-arithmeticity in the i.i.d case and for Markov chains that are almost onto (see section 7.1).

We assume that  $E(X_1) = 0$  and that

$$\Sigma := \{x | \mathbb{P}(S_n = x) > 0 \text{ for some } n\} = \mathbb{Z}.$$

Otherwise, since  $\Sigma$  is a subgroup of  $\mathbb{Z}$ , in the recurrent case, we'll need to relabel the state space.

Let

$$p = \inf \{k \in \mathbb{N} : \mathbb{P}(S_k = 0) > 0\}$$

be the period of the random walk. It follows that

$$\left|E\left(e^{itX}\right)\right| = 1 \Leftrightarrow t \in \frac{2\pi}{p}\mathbb{Z}$$

In this case the random variable  $X_1$  takes values in a proper coset of  $\mathbb{Z}$ . The periodic structure of  $S_n$  is that for  $n \in \mathbb{N}$  and  $k \in \{1, \dots, p-1\}$ ,  $S_{np+k}$  takes values in one of the  $p-1$  cosets of  $\mathbb{Z}$  of the form  $p\mathbb{Z} + j$ ,  $j = 0, \dots, p-1$ .

In this case the potential kernel estimate (Lemma 5) should be corrected to state that for every  $x, y \in \mathbb{Z}$ ,

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{p-1} \mathbb{P}(S_{np+k} = x) - \mathbb{P}(S_{np+k} = y) \right| \leq C|x-y|, \quad (7.1)$$

which is a special case of the corollary from lemma 7 in [8]. Lemma 4 also holds in this case (see Lemma 2 in [8]). These two ingredients are enough to prove the fourth moment inequality, Lemma 6. To see how this is done notice that (here we keep the notation of Lemma 6)

$$E\left(\mathbf{1}_{\xi_1}(S_{i_1}) \psi(S_{i_2}) \mathbf{1}_{\xi_2}(S_{i_3}) \psi(S_{i_4})\right) = [\mathbb{P}(S_{i_1} = \xi_1) \cdot F(i_1, i_2, i_3) \cdot (\mathbb{P}(S_{i_4-i_3} = x) - \mathbb{P}(S_{i_4-i_3} = y))] \quad (7.2)$$

where

$$F(i_1, i_2, i_3) = [\mathbb{P}(S_{i_2-i_1} = x - \xi_1) \mathbb{P}(S_{i_3-i_2} = \xi_2 - x) \quad (7.3)$$

$$- \mathbb{P}(S_{i_2-i_1} = y - \xi_1) \mathbb{P}(S_{i_3-i_2} = \xi_2 - y)] \quad (7.4)$$

$$= \mathbb{P}(S_{i_2-i_1} = x - \xi_1) [\mathbb{P}(S_{i_3-i_2} = \xi_2 - x) - \mathbb{P}(S_{i_3-i_2} = \xi_2 - y)] \\ + \mathbb{P}(S_{i_3-i_2} = \xi_2 - y) [\mathbb{P}(S_{i_2-i_1} = x - \xi_1) - \mathbb{P}(S_{i_2-i_1} = y - \xi_1)]$$

Therefore, since

$$E^\mu (L_n(x) - L_n(y))^4 = \sum_{\vec{\xi}} \sum_{\vec{i}} E^\mu (\mathbf{1}_{\xi_1}(S_{i_1}) \psi(S_{i_2}) \mathbf{1}_{\xi_2}(S_{i_3}) \psi(S_{i_4})),$$

from (7.2) and (7.3) it follows that we need to estimate two sums. The first is

$$\sum_{i_1=1}^n \left[ \mathbb{P}(S_{i_1} = \xi_1) \sum_{i_2=0}^{n-i_1} \mathbb{P}(S_{i_2} = x - \xi_1) \sum_{i_3=0}^{n-i_1-i_2} \{ \mathbb{P}(S_{i_3} = \xi_2 - x) - \mathbb{P}(S_{i_3} = \xi_2 - y) \} \right. \\ \left. \sum_{i_4=0}^{n-i_1-i_2-i_3} \{ \mathbb{P}(S_{i_4} = x) - \mathbb{P}(S_{i_4} = y) \} \right]. \quad (7.5)$$

In order to bound this term we first notice that

$$\sum_{i_4=0}^{n-i_1-i_2-i_3} \{ \mathbb{P}(S_{i_4} = x) - \mathbb{P}(S_{i_4} = y) \} = \sum_{l=0}^{\lfloor (n-i_1-i_2-i_3)/p \rfloor} \sum_{j=0}^{p-1} \{ \mathbb{P}(S_{l \cdot p + j} = x) - \mathbb{P}(S_{l \cdot p + j} = y) \} \pm p.$$

here  $a = b \pm c$  means  $|a - b| \leq c$ . We do a similar rearrangement to the sum of  $i_3$ . It then follows that the term in (7.5) is strictly smaller than

$$\sum_{i_1=1}^n \left[ \mathbb{P}(S_{i_1} = \xi_1) \sum_{i_2=0}^{n-i_1} \mathbb{P}(S_{i_2} = x - \xi_1) \cdot \left( p + \sum_{l_1=0}^{\infty} \left| \sum_{j_1=0}^{p-1} \{ \mathbb{P}(S_{i_3} = \xi_2 - x) - \mathbb{P}(S_{i_3} = \xi_2 - y) \} \right| \right) \right. \\ \left. \cdot \left( p + \sum_{l=0}^{\infty} \left| \sum_{j=0}^{p-1} \{ \mathbb{P}(S_{l \cdot p + j} = x) - \mathbb{P}(S_{l \cdot p + j} = y) \} \right| \right) \right].$$

The conclusion follows from a double application of the inequality (7.1) and the bound in Lemma 2 of [8]. The second term is dealt with similarly and the rest of the proof of Borodin's theorem remains the same. Thus we have:

**Theorem 4** *Let  $X_1, X_2, \dots, X_n, \dots$  be an i.i.d sequence with  $E(X_1) = 0$  and  $E(X_1^2) = 1$  such that the random walk  $S_n$  is recurrent. Then there exists a probability space  $(X', \mathcal{B}', \mathbb{P}')$  and processes  $W'_n, W'_\sigma : X \rightarrow D_+$  such that:*

1.  $W_n \stackrel{d}{=} W'_n; W \stackrel{d}{=} W'$ .
2. *With probability one  $W'_n$  converges to  $W'$  uniformly on compact subsets of  $[0, \infty)$ .*
3. *For every  $\epsilon, T > 0$  the processes  $l'_n(t, x)$  and  $l'(t, x)$  defined with respect to  $W'_n$  and  $W'_\sigma$  satisfy the relationship:*

$$\lim_{n \rightarrow \infty} \mathbb{P}' \left( \sup_{t \in [0, T], x \in \mathbb{R}} |l'_n(t, x) - l'(t, x)| > \epsilon \right) = 0.$$

### 7.1 The periodic case of finite state Markov chains

To drop the assumption of strong aperiodicity when  $X_1, \dots, X_n, \dots$  is a Markov chain we use the dynamical setting introduced in section 1. Let  $P : S \times S \rightarrow [0, 1]$  be the transition matrix,  $\mu$  be an initial distribution on  $S$  and  $P^\mu$  the

probability measure on  $S^{\mathbb{N}}$  generated by  $\mu$  and  $P$ . We then look at the Markov shift  $(S^{\mathbb{N}}, \mathcal{B}, P^\mu, T)$  where  $T$  is the shift and  $\mathcal{B}$  is the Borel  $\sigma$ -algebra generated by the cylinder sets

$$[s_0 s_1 \dots s_n] = \left\{ \omega \in S^{\mathbb{N}} : \omega_i = s_i, i = 0, 1, \dots, n \right\}.$$

In this case  $X_n = f(T^n \omega) = \omega_n$  and if the function  $f$  is periodic there exists a solution to

$$e^{itf(\omega)} = \lambda \frac{\varphi(\omega)}{\varphi(T\omega)}$$

where either  $t \notin 2\pi\mathbb{Z}$  or  $\varphi$  is not constant. To make the same analysis as for random walks we need to exclude the case  $\varphi \neq \text{const}$  (then summation over the Markov Chain will yield the same periodic structure as in the i.i.d case). A sufficient condition for that is that the system  $(X, \mathcal{B}, P_\mu, T, \alpha)$  is almost onto with respect to the partition  $\alpha = \{[s_0] : s_0 \in S\}$ , meaning that for every  $a, b \in \alpha$  there exist sets  $a_0, \dots, a_n$  such that  $a_0 = b$ ,  $a_n = c$  and  $Ta_k \cap Ta_{k+1} \neq \emptyset$  (for details see [2, Section 3]).

## 8 Applications to complexity of random walks in random sceneries with a Markov chain base

A Random Walk in Random Scenery is a skew product probability preserving transformation which is defined as follows:

The *random scenery* is an invertible probability preserving transformation  $(Y, \mathcal{C}, \nu, S)$  and the *random walk in random scenery*  $(Y, \mathcal{C}, \nu, S)$  with jump random variable  $\xi$  (assumed  $\mathbb{Z}$ -valued) is the skew product  $(Z, \mathcal{B}(Z), m, T)$  defined by

$$Z := \Omega \times Y, \quad m := \mu_\xi \times \nu, \quad T(w, y) = (Rw, S^{w_0} y)$$

where

$$(\Omega, \mathcal{B}(\Omega), \mu_\xi, R) = \left( \mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}}), \prod \text{dist} \xi, \text{shift} \right)$$

is the shift of the independent jump random variables.

Aaronson [1], assuming  $\xi$  is in the domain of attraction of an  $\alpha$ -stable law with  $\alpha > 0$ , has studied the relative complexity and the relative entropy dimension of  $T$  over the base. We refer the reader to [1] for the definitions.

One can ask if Aaronson's result [1, Theorem 3, Section 1] remain true if we change the base  $(\Omega, \mathcal{B}(\Omega), \mu_\xi, R)$  to a strongly aperiodic, irreducible, finite state Markov Shift  $(\Omega, \mathcal{B}(\Omega), P^\mu, R)$ .

With Theorem 1 at hand, Aaronson's proof, which can even be simplified since the Brownian motion is a.s continuous, can be carried out verbatim.

## 9 Appendix

**Proposition 3** *Let  $(X, \mathcal{B}, \mu)$  be a probability space,  $f : X \rightarrow \mathbb{S}^1$  (the unit sphere) measurable. Then*

$$\left| \int f(x) d\mu \right| = 1$$

implies  $f = \text{const.}$

*Proof* Assume that  $f$  is not constant. Then there are two disjoint intervals  $I_1, I_2 \subseteq \mathbb{T}$  such that  $\mu(f^{-1}(I_1)) > 0$  and  $\mu(f^{-1}(I_2)) > 0$ . It follows that

$$\left| \int_X f(x) d\mu \right| \leq \left| \int_{f^{-1}(I_1)} f(x) d\mu + \int_{f^{-1}(I_2)} f(x) d\mu \right| + \mu\left(X \setminus \left(f^{-1}(I_1) \cup f^{-1}(I_2)\right)\right) < 1.$$

This is true since if for  $j \in \{1, 2\}$

$$\left| \frac{1}{\mu(f^{-1}(I_j))} \int_{f^{-1}(I_j)} f(x) d\mu \right| < 1$$

then it is trivial, else

$$\frac{1}{\mu(f^{-1}(I_j))} \int_{f^{-1}(I_j)} f(x) d\mu \in I_j, \quad j = 1, 2$$

and the mean of two different values on the unit sphere is of modulus strictly less than 1.

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